TOPOLOGY - III, SOLUTION SHEET 13

Exercise 1. We refer the reader to exercise 1 of sheet 11 for the CW complex structures.

(1) We can realise S^n as a quotient of D^n, by identifying ∂D^n to a point. Therefore S^n has a cell in dimensions 0 and n. Therefore we have that the cellular complex for S^n is:
$$0 \to \mathbb{Z} \to 0 \to ... \to \mathbb{Z} \to 0.$$
With the $\mathbb{Z}$ in degrees $n, 0$. Note that all the boundary maps are the 0 maps, even when $n = 1$ (the case for S^1 is explained in part 6, along with the general case of connected graphs). Therefore we obtain that $H_k(S^n) = \mathbb{Z}$ if $n = k$ or 0 and 0 otherwise.

(2) Please refer to example 2.36 on page 141 of Hatcher's book.

(3) Please refer to example 2.37 on page 141 of Hatcher's book.

(4) Please refer to example 2.42 on page 144 of Hatcher's book.

(5) Since $\mathbb{CP}^n$ has exactly one cell in all even dimensions and no cells in odd dimensions we have that the cellular complex for $\mathbb{CP}^n$ is:
$$0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to ... \to \mathbb{Z} \to 0 \to 0.$$
Where the $\mathbb{Z}$ live in even degrees. Therefore we obtain that $H_k(\mathbb{CP}^n)$ is equal to $\mathbb{Z}$ in degrees $k = 0, 2, ..., n$ and equal to 0 otherwise.

(6) Let E be the edge set of a graph Γ and I be the vertex set. Then the cellular complex for Γ is:
$$0 \to \mathbb{Z}^{|E|} \xrightarrow{\sigma} \mathbb{Z}^{|I|} \to 0.$$
Moreover if $[p, q]$ is an edge with end vertices p and q then $\sigma([p, q]) = p - q$. The orientation of the edges is determined by the choice of attaching maps $\varphi_\alpha : S^0_\alpha = \{p, q\} \to I$. Note that $Im\, \sigma$ is free of rank $|I| - 1$ and hence $H_1(\Gamma) = Ker\, \sigma$ is a free abelian group of rank $|E| - |I| + 1$. Since Γ is connected, we also have that $H_0(\Gamma) = \mathbb{Z}$.

Exercise 2. We refer the reader to section 2 of the notes by Jens Hemelaer, on Schubert cells which are uploaded on the moodle.